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# 6 The Structuralist View of Measurement: An Extension of Received Measurement Theories

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There is a vast literature on measurement and on the relation to theory of measurement, experience, evidence.<sup>2</sup> With few exceptions, these accounts have cases of simple, isolated, empirical, numerical hypotheses, like the ideal gas law, Ohm's law or Pythagoras' theorem, as their paradigm examples from which the general ideas are drawn and against which they are checked. However, scientific progress leads from those isolated beginnings to comprehensive networks of theories in which the systems originally studied form only small fragments. New devices of measurement are then introduced, and many (or even most) of the original methods of measurement are regarded as obsolete after a while. Such new devices of measurement often are quite different from the original methods. For instance, they often are "dependent" on the very same theory whose functions they determine—an idea hardly consistent with traditional concepts of measurement. The broader perspective of comprehensive theoretical networks and the practice of measurement in such broader contexts call for an extension of traditional views about measurement—an extension that may well necessitate some revision.

The term "measurement" in ordinary as well as in current scientific language is ambiguous. Sometimes counting is included, sometimes numerical calculations are excluded, sometimes concatenation, and sometimes experiment are regarded as necessary. A theory of measurement cannot perfectly match all these

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<sup>1</sup>Much preliminary work from which the structuralist view of measurement presented here finally developed was done in DFG project Ba 678/1 and Ba 678/2. I am indebted to B. Lauth, R. Niederée, and R. D. Luce for critical comments and suggestions on the first draft.

<sup>2</sup>For a selection, see the references in Berka (1983) and Krantz et al. (1971).

usages, simply because they are inconsistent. I will not be concerned here with defending my special use of the word measurement. Rather I am interested in a theory about those kinds of scientific practice that aim at the determination of particular features, properties, relations attached to special, prepared objects or events. These kinds of scientific practice provide an array of phenomena rich enough to justify metascientific theorizing—maybe even too rich, given the lack of success up to now—and they include the phenomena studied by traditional theories of measurement.

This chapter is intended to sketch the basis for an extended concept of measurement that includes measurement in the context of big nets of theories as well as more “local,” “fundamental,” traditional situations. I cannot elaborate on the details and applications of this extended account; there are too many for this chapter.<sup>3</sup> Instead I will concentrate on the way this account relates to and extends traditional approaches.

I apologize for throwing together much outstanding and original work on measurement in one big pot to which the label “received view” is attributed, but I see no other possibility to arrive at a condensed, general comparison. Of course the particular label is chosen on purpose, although with a positive, “conservative” touch: What we received is a great heritage forming a solid *fundament* on which we can further build.

## THE RECEIVED VIEW OF MEASUREMENT

### Overview

According to the received view, measurement consists, very roughly, of an assignment of numbers to concrete, empirical objects or events such that the assigned numbers (together with suitable standard mathematical operations) *represent* the empirical objects or events (together with the empirical operations defined among them). In other words, for a given *empirical structure* consisting of a domain of concrete objects or events and concrete operations among these, measurement in a piecemeal fashion, establishes a mathematical structure consisting of a domain of numbers and abstract operations among these such that the mathematical structure represents the empirical one, which means that both structures are homomorphic or even isomorphic under a suitable mapping of empirical objects to mathematical objects. Often the empirical structure can be characterized by axioms expressing operationally testable propositions, and the existence of a homomorphism into a suitable mathematical structure can be proved from these axioms. The statement expressing the existence and uniqueness of such a homomorphism is then called a *representation theorem*, and the mapping

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<sup>3</sup>Compare Balzer (1983, 1985b).

from objects to numbers (or the whole class of these) is termed a *scale*. Among the empirical operations there will usually be a relation of order, and classes of such orderings (subject to further conditions) are called *quantities*. I will say that the values of a numerical function *belong to* a quantity, if the function represents empirical structures of which the quantity's orderings are parts.

Historically this view may be traced back into antiquity. Euclid's axioms may be regarded as operationally testable statements about unspecified concrete operations. That is, they become testable once suitable operations are specified. In fact, these axioms miss only little of the axioms in D3 below. In modern times, things got started again when Hoelder produced the first mathematically sound representation theorem (Hölder, 1901), and this spread into applications of empirical sciences—most prominently of psychology. The classical modern formulation was achieved in *Foundations of measurement* by Krantz, Luce, Suppes, and Tversky (1971), which expresses a strong, progressive research program. Recent developments have focused on the study of interlocked concatenation, conjoint systems, and a generalized notion of scale type (e.g., Luce and Narens, 1985; Narens, 1985). I cannot do justice here to all the different individual contributions and expressions of the approach.

In spite of individual differences, all (or most) authors seem to agree on the distinctive feature that measurement yields numerical representations of empirical structures. For this reason, the received view of measurement is usually called the *representationalist* view.

This view of measurement is confirmed along several lines. First, numbers or mathematical objects allow for many manipulations that are impossible or at least difficult and inefficient to perform with concrete objects or events: Numbers can be compared according to their magnitude, and they can be added, multiplied, raised to the power, and so on. All this can be done without worrying about dimensions, about whether different numbers belong to the same quantity, and whether it makes sense to, say, add numbers belonging to different quantities. In this sense, numerical representations are an efficient means to unify and to make coherent scientific talk about and scientific practise dealing with concrete objects, events, and operations. Second, in several cases, historical developments in fact led to the establishment of numerical representations. This holds for geometry, chronometry, kinematics, gravitational and inertial mass and temperature, to mention some important cases in which numerical representation is crucial.<sup>4</sup> Third, it may be pointed out that numbers, after all, *are nothing but* "numerical representations" of empirical structures. So the representation of the latter in terms of numbers has a strong analytic flavor.<sup>5</sup>

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<sup>4</sup>Other examples for which representation theorems can be proved but for which the empirical structures are less reliable are found, for instance, in Krantz et al. (1971).

<sup>5</sup>This view I share with perhaps few philosophers of mathematics. See Balzer (1979).

## Fundamental Measurement

Fundamental measurement begins at the individual level of concrete systems. Let us look at one such system that conceptualizes or captures just one single process of measurement in the course of which one value, to be called the *measured value* in the following, is determined. A paradigm is measurement of distance by means of rigid rods, and the measurement of one distance-value is conceptualized as a process in which different copies of a unit-rod are put together (concatenated) until the two points in question can be connected by the concatenated sequence of units. The distance is then “read off” by counting the minimal number of unit rods necessary in order to establish a connection.

In general, such a system will involve concrete *objects* or *events* (like points marked on rigid bodies in the distance example, or processes in the measurement of time) as well as concrete operations among those objects or events (like concatenation and comparison of rigid rods). The operations can be classified into a certain, small number of *few* kinds, and by looking at many similar systems, several properties of the kinds of operations can be extracted and stated explicitly as axioms about the kinds of operations involved. These items, the *domain*  $D$  of objects, the *kinds of concrete operations*  $\circ_1, \dots, \circ_r$  involved, and the *axioms* characterizing these operations make up what is called an *empirical structure*. This is a structure  $\langle D, \circ_1, \dots, \circ_r \rangle$  satisfying the axioms.

However, in order to speak of measurement, representation is necessary. The empirical structure we find in some real system is matched by a *numerical structure*  $\langle \mathbb{R}, m_1, \dots, m_r \rangle$  where  $\mathbb{R}$  is the set of real numbers, and each  $m_i$  ( $i \leq r$ ) is a relation of the same type over  $\mathbb{R}$  as is  $\circ_i$  over  $D$ . The numerical structure  $\langle \mathbb{R}, m_1, \dots, m_r \rangle$  is said to *represent* the empirical structure  $\langle D, \circ_1, \dots, \circ_r \rangle$ , iff there is a mapping  $\omega: D \rightarrow \mathbb{R}$ , which is a homomorphism with respect to the relations on both sides. Altogether the conceptualization of a concrete system as a system capturing some process of measurement yields an entity of the following form:

$$\langle \langle D, \circ_1, \dots, \circ_r \rangle, \langle \mathbb{R}, m_1, \dots, m_r \rangle, \omega \rangle$$

(i.e., an empirical structure together with a numerical structure and some homomorphism  $\omega$  between both). Let us call any such entity a *system of fundamental measurement*.

Here is a simple example.<sup>6</sup>

- D1** (a).  $x$  is a *model of length measurement with units*  $U$  iff there exist  $D, \lesssim, \circ$  and  $\omega$  such that  $x = \langle \langle D, \lesssim, \circ \rangle, \langle \mathbb{R}, \leq, + \rangle, \omega \rangle$  and the following holds:

1.  $D$  is a set and  $\emptyset \neq U \subseteq D$ .
2.  $\lesssim \subseteq D \times D$  is transitive, reflexive, and connected.

<sup>6</sup>In D1 and D3 below and the corresponding proofs, I use the following notation: “ $a \equiv b$ ” for “ $a \lesssim b$  and  $b \lesssim a$ ,” “ $a < b$ ” for “ $a \lesssim b$  and not  $b \lesssim a$ ,” “ $x(\omega')$ ” for the result of substituting  $\omega'$  in  $x$  for  $\omega$ , and “ $na$ ” for “ $a \circ \dots \circ a$ ” ( $n$ -times).  $\mathbb{R}^+$  and  $\mathbb{N}$  denote the sets of positive reals and of natural numbers respectively.

3.  $\circ$  is a partial function from  $D \times D$  to  $D$  and is associative.
4. For all  $a, b, c \in D$  for which  $a \circ b, a \circ c, b \circ c$  are defined:  
 $a < a \circ b$ , and  $(a \lesssim b, \text{ iff } a \circ c \lesssim b \circ c, \text{ iff } c \circ a \lesssim c \circ b)$
5. For all  $b, b' \in U$ :  $b \equiv b'$ .
6. For all  $a \in D$ , there exist  $b_1, \dots, b_n \in U$  such that  $b_1 \circ \dots \circ b_n$  is defined, and  $b_1 \circ \dots \circ b_n \equiv a$ .
7.  $\omega: D \rightarrow \mathbb{R}$  and the following holds for all  $a, b \in D$ :
  - 7.1.  $a \lesssim b$ , iff  $\omega(a) \leq \omega(b)$ .
  - 7.2.  $\omega(a \circ b) = \omega(a) + \omega(b)$  (when  $a \circ b$  is defined).
- (b).  $x$  is a *model of length measurement* iff there exists some  $U$  such that  $x$  is a model of length measurement with units  $U$ .
- T1** (a). If  $x$  is a model of length measurement with units  $U$  and with units  $U'$ , then for all  $b, b'$ : if  $b \in U$  and  $b' \in U'$  then  $b \equiv b'$ .
- (b). If  $x = \langle \langle D, \lesssim, \circ \rangle, \langle \mathbb{R}, \leq, + \rangle, \omega \rangle$  and  $x' = \langle \langle D, \lesssim, \circ \rangle, \langle \mathbb{R}, \leq, + \rangle, \omega' \rangle$  are models of length measurement, then there exists  $\alpha \in \mathbb{R}^+$  such that for all  $a \in D$ :  $\omega(a) + \alpha \cdot \omega'(a)$ .
- (c). If  $\langle D, \lesssim, \circ \rangle$  and  $U$  are such that D1a1 to 6 are satisfied, then there exists  $\omega: D \rightarrow \mathbb{R}$  such that a-7 is satisfied and  $\omega$  is uniquely determined up to some  $\alpha \in \mathbb{R}^+$ .

The proofs of this and the other theorems are given in the appendix.

Of course, by considering just one, single, real system, one hardly would end up with a system of fundamental measurement. In reality, many different but similar systems exist and are taken into consideration. Only on the basis of a large number of similar systems or situations is it possible to extract axioms for empirical structures that express observed regularities. The natural basis for the description of fundamental measurement, therefore, is given not by single systems but rather by classes of such systems. I will call them *fundamental measurement classes*.

- D2**  $X$  is a *fundamental measurement class* (with respect to transformations of type  $\tau$ ), iff there is a set-theoretic formula  $F$  such that the following holds:
1.  $F$  can be validated in structures of the form  $\langle \langle D, \circ_1, \dots, \circ_r \rangle, \langle \mathbb{R}, m_1, \dots, m_r \rangle, \omega \rangle$  and  $F$  expresses (among other things) empirical regularities about  $\langle D, \circ_1, \dots, \circ_r \rangle$ .
  2.  $X$  is the class of all structures in which  $F$  is valid.
  3. For all  $D, \circ_1, \dots, \circ_r, m_1, \dots, m_r, \omega$ , if  $F(D, \circ_1, \dots, \circ_r, m_1, \dots, m_r, \omega)$ , then the following holds:
    - 3.1.  $\circ_1, \dots, \circ_r$  can be interpreted by concrete operations.
    - 3.2.  $\omega$  is a homomorphism from  $\langle D, \circ_1, \dots, \circ_r \rangle$  to  $\langle \mathbb{R}, m_1, \dots, m_r \rangle$ .
    - 3.3.  $\omega$  is uniquely determined up to transformations of type  $\tau$  (that is, every other homomorphism from  $\langle D, \dots, \circ_r \rangle$  to  $\langle \mathbb{R}, \dots, m_r \rangle$  is obtained from  $\omega$  by some transformation of type  $\tau$ ).

Condition (3.3) may seem redundant to some representationalists, because it usually follows as a theorem from appropriate axioms about  $\sigma_1, \dots, \sigma_r$ . However, in the general case, there may be a considerable multiplicity of scales  $\omega$  satisfying the other requirements of D2, that is,  $\omega$  may be determined only up to rather general transformations. If we imagine a system of fundamental measurement to describe some real process of measurement, we expect specific numerical values to be produced. However, if  $\omega$  is determined only up to very general transformations (like monotonic transformations), we will not get definite values. Some authors would still speak of measurement in such a case. What matters for them is representation, not determination. However, in practice, determination is most important, and D2 (3.3) may help to keep us aware of that.

Fundamental measurement classes have further interesting properties. Let us write  $D^x$ ,  $o^x$ ,  $\omega^x$ , and so on for the various components of a system  $x = \langle \langle D, o, \dots, \omega \rangle \rangle$  and also  $U^x$  for the set of units occurring in D1. For a given fundamental measurement class  $X$ , let us define the *join* of  $X$ ,  $\cup X$ , as the structure  $\langle \bigcup_{x \in X} D^x, \bigcup_{x \in X} o^x, \dots, \bigcup_{x \in X} \omega^x \rangle$ . On the operational level, the join,

of, say, two systems may simply be a conceptual artifact (think of two distance measurements in completely disjoint systems), or it may be an extension of the joined systems (e.g., if we measure by rigid rods the distance from  $e_1$  to  $e_2$  in one system  $s_1$ ; from  $e_2$  to  $e_3$  in a second system  $s_2$ ; and from  $e_1$  to  $e_3$  in the system  $s_3$  obtained by joining  $s_1$  and  $s_2$ ). On the numerical level, the join of two systems yields an additional constraint on the choice of representing numbers; only those assignments are admissible that assign the same numbers to objects occurring simultaneously in different systems (the distance between  $e_1$  and  $e_2$  in  $s_1$ , for instance, must be the same as between  $e_1$  and  $e_2$  in the extended system  $s_3$ ).

Formally such a constraint can be stated as a property of a fundamental measurement class  $X$ .  $X$  satisfies the constraint iff, for any two systems,  $x = \langle \langle D, \dots \rangle, \dots, \omega \rangle$ ,  $x' = \langle \langle D', \dots \rangle, \dots, \omega' \rangle \in X$ , the following holds: For all objects  $a \in D \cap D'$ , the representing numbers  $\omega(a)$  and  $\omega'(a)$  with respect to  $x$  and  $x'$  are the same— $\omega(a) = \omega'(a)$ . In other words, the numerical representation  $\omega(a)$  of an object  $a$  in a system  $x$  must not change when  $a$  is considered from the point of view of some other system  $x'$ . In measurement of distances, this amounts, among other things, to the requirement that some unit-rod in one system should also be a unit-rod in any other system. Expressed still differently, the constraint says that, in the join of any two systems  $x, z \in X$ , the numerical assignment  $\omega^x \cup \omega^z$  in fact is a function:  $\omega^x \cup \omega^z: D^x \cup D^z \rightarrow \mathbb{R}$ . Similarly if we take the join of all of  $X$ ,  $\bigcup X$ , we still have a function  $\bigcup_{x \in X} \omega^x: \bigcup_{x \in X} D^x \rightarrow \mathbb{R}$ .<sup>7</sup>

Usually the join  $\cup X$  of a fundamental measurement class will not capture any

<sup>7</sup>See Balzer (1978), for applications of this kind and of other kinds of constraints to the case of geometry.

concrete process of measurement. Rather it captures a whole domain of objects to which some given measurement procedure can be applied. In the technical literature, such idealized universal systems play a dominant role for the standard, elegant representation theorems can be proved only for the idealized systems.<sup>8</sup> As a typical example, let us just cite the closed extensive positive systems of Krantz et al. (1971, p. 73).

**D3** (a).  $x$  is a *closed extensive positive structure* iff there exist  $D, \lesssim, \circ$  such that  $x = \langle D, \lesssim, \circ \rangle$  and the following holds:

1.  $D$  is a nonempty set.
2.  $\lesssim \subseteq D \times D$  is transitive, reflexive, and connected.
3.  $\circ: D \times D \rightarrow D$  is associative.
4. For all  $a, b, c \in D$ :  $a < a \circ b$ , and  $(a \lesssim b \text{ iff } a \circ c \lesssim b \circ c \text{ iff } c \circ a \lesssim c \circ b)$ .
5. For all  $a, b, c, d \in D$ : if  $a < b$ , then there is  $n \in \mathbb{N}$  such that  $na \circ c \lesssim nb \circ d$ .

(b).  $x$  is an *idealized model of length measurement* iff there exist  $D, \lesssim, \circ$  and  $\omega$  such that  $x = \langle \langle D, \lesssim, \circ \rangle, \langle \mathbb{R}, \leq, + \rangle, \omega \rangle$  and the following holds:

1.  $\langle D, \lesssim, \circ \rangle$  is a closed extensive positive structure.
2.  $\omega: D \rightarrow \mathbb{R}$ .
3. For all  $a, b \in D$ :  $(a \lesssim b \text{ iff } \omega(a) \leq \omega(b))$  and  $\omega(a \circ b) = \omega(a) + \omega(b)$ .

**T2** (a). If  $\langle D, \lesssim, \circ \rangle$  is a closed extensive positive structure, then there exists  $\omega$  such that  $\langle \langle D, \lesssim, \circ \rangle, \langle \mathbb{R}, \leq, + \rangle, \omega \rangle$  is an idealized model of length measurement and  $\omega$  is determined uniquely up to some  $\alpha \in \mathbb{R}^+$ .

(b). If  $X$  is a class of models for length measurement such that the following holds: (i) for all  $x, u \in X$ , there exists a common extension<sup>9</sup>  $z \in X$ ; (ii) for all  $x \in X$  and all  $a, b \in D$  such that  $a < b$ , there is some  $u \in X$  such that  $u$  is an extension of  $x$ ; and there is some  $n \in \mathbb{N}$  such that, in  $u$ ,  $b \lesssim na$ ; (iii)  $\bigcup_{x \in X} \sigma^x$  is a total function, then  $\bigcup X$  is an idealized model of length measurement.

Clearly closed extensive positive structures do not capture single processes of measurement. They are most naturally regarded as the join of an appropriate fundamental measurement class. In T2, some rather strong sufficient conditions for this are stated.<sup>10</sup>

<sup>8</sup>There are less elegant systems that apply also to unidealized finite systems, like the one in D1 above, or the one in Krantz et al. (1971, p. 103).

<sup>9</sup>That  $z$  is an *extension* of  $x$  means that  $D^x \subseteq D^z$ , and  $\lesssim^x$  and  $\sigma^x$  are the restrictions of  $\lesssim^z$  and  $\sigma^z$  to  $D^x$  respectively.

<sup>10</sup>It seems to be worthwhile studying this case and looking for weaker conditions that are necessary too.



## Derived Measurement and the Structure of Science

Fundamental measurement occurs in science whenever a new range of phenomena for the first time is approached in a quantitative way. However, when one or several theories are established in some domain, new concepts of a theoretical nature will be introduced in order to get a more efficient organization of the available data. Often such new concepts will not be accessible to fundamental measurement in a straightforward way; they are determined by *derived* measurement. Derived measurement of a function simply consists of an explicit definition of this function in terms of other functions that are accessible to fundamental measurement, of measuring appropriate values of these latter functions, and of calculating the desired value(s) of the defined function from the measured values according to the given definition.

Consider the trivial example of velocity. The quantitative notion of velocity was introduced after quantitative concepts of spatial and temporal distance had developed. Velocity cannot be measured fundamentally—at least not in any straightforward way—and the same holds for mean velocity. The first step in the development of the notion was to define mean velocity as spatial distance travelled divided by time needed. In order to measure mean velocity, one has to measure (say, fundamentally) spatial and temporal distances and to apply the definition to the values, thus, obtained.

This account yields a nice and precise picture of the general structure of science: “Basic” terms, which can be fundamentally measured, are introduced in the beginning, and the vocabulary of science is enlarged step-by-step by introducing new terms definable by those that are antecedently available or by introducing new terms that can be fundamentally measured. Nice though it may be, this picture does not show how science actually develops. Usually new concepts are not explicitly defined, the previous example of velocity being one of the rare exceptions.<sup>11</sup> The discussions about theoretical terms in logical empiricism, as well as recent formal results about theoreticity<sup>12</sup> provide sufficient evidence here.

Recent work on measurement in conjoint systems<sup>13</sup> suggests that the question of definability in connection with derived measurement may be less central. The procedure just outlined might be replaced by some fundamental method directly producing the “defined” quantity; under suitable conditions, this direct representation can be decomposed multiplicatively. At present, there are only few, real-life examples of conjoint measurement, and these are mainly initiated by the development of the abstract notion. Still one could insist that derived measurement in principle might be replaced by conjoint measurement and that the dis-

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<sup>11</sup>A more typical example is found in D4 below where the condition eligible for a definition (D4-a-5) does not guarantee uniqueness.

<sup>12</sup>See Balzer (1985a, 1986) and Gähde (1983).

<sup>13</sup>Compare, for instance, Luce & Narens (1985), Narens & Luce (1986).

tance between scientific practice and the picture emerging from derived measurement, therefore, is artificial. However, replacing derived measurement by conjoint measurement would not bring us closer to scientific practise. In the natural sciences, there is a large class of methods of measurement that simply are not of the conjoint type.

In any case, the picture of the structure of science emerging from the representationalist view is not the most adequate one, and my claim is that the structuralist view of measurement provides a more adequate one.

## THE STRUCTURALIST VIEW OF MEASUREMENT

### Overview

According to the structuralist view, measurement consists in determining a *datum* (function value, truth value), which, on a regular pattern, is uniquely given by other, known, or available data. If all the data are represented as values of real valued functions, the regular pattern will usually be represented by a mathematical equation, and the determination of the value one is looking for (the measured value) amounts to its calculation from other, presupposed values by “solving the equation.”

Let us start analyzing this picture at the level of single systems—as in the first case. What we have before us is a concrete process of measurement that, after some suitable preparations, consists of the development of a real system until the measured value is produced. Such a system is distinguished from its environment as sharply as we are able to separate different systems in different situations—an ability that we may safely assume in the present context. Further, it usually is not too difficult to determine two instants marking the “beginning” and the “end” of the process of measurement (which need not be unique). If we start to conceptualize the system that is given by the process, we see that there are already some (one or several, but few) theoretical pictures, *theories*, that can be used for the conceptualization. We assume that the system at hand is a model of that theory or theories. Such a model *capturing* or *conceptualizing* or *describing* a real system that is given by some process of measurement I call a *measuring model*. Clearly, if a process of measurement involves more than one theory for its description, we may analyze it and deal with various subprocesses, each of which yields just one model of a theory (i.e., just one measuring model). I will not go into the question here of how to put together the various measuring models, thus, obtained in order to provide a complete description of the original process; in the following, I will deal only with processes that are governed or can be described by just one theory.<sup>14</sup>

As an example, let us consider the measurement of mass by means of colli-

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<sup>14</sup>Compare Balzer (1985b) and Balzer and Wollmershaeuser (1986), for studies of more complex situations.

sions.<sup>15</sup> Consider three particles colliding with each other at one instant. Their velocities before and after the collision are approximately constant and are measured by suitable means. Furthermore, let us assume that the velocity vectors have some special geometrical configuration: the vectors of velocity differences (before and after the collision, for each particle) are such that it is not possible to have a plane passing through the origin of the three vectors and all three vectors lying on one side of the plane—D4(a)(6) below. In this situation, the law of conservation of total momentum—which is the central axiom of collision mechanics—allows us to calculate the particles' mass-ratios from their velocities.<sup>16</sup>

**D4** (a).  $x$  is a *measuring model for mass by collisions* (with respect to  $p_1$ ) iff there exist  $P, T, v, m$  such that  $x = \langle P, T, \mathbb{R}, v, m \rangle$  and the following holds:

1.  $P$  is a three-element set (of "particles") and  $p_1 \in P$  (we write  $P = \{p_1, p_2, p_3\}$ ).
2.  $T = \{b, a\}$  is a two-element set ("before" and "after").
3.  $v: P \times T \rightarrow \mathbb{R}^3$  (velocity function).
4.  $m: P \rightarrow \mathbb{R}^+$  (mass function).
5.  $\sum_{p \in P} m(p)v(p, b) = \sum_{p \in P} m(p)v(p, a)$   
(law of conservation of momentum).
6. the subspace of  $\mathbb{R}^3$  generated by  $w_1, w_2, w_3$  has two dimensions and there is no  $u \in \mathbb{R}^3$  such that the following holds:
  - 6.1. For all  $i \in \{1, 2, 3\}$ :  $w * u \geq 0$ .
  - 6.2. There is some  $i \in \{1, 2, 3\}$  such that  $w * u > 0$ .
7.  $m(p_1) = 1$ .

(b).  $x$  is a *measuring model for mass-ratios by collision* iff  $x = \langle P, T, \mathbb{R}, v, m \rangle$  is, as in part a) and D4 (a) (1) to (a) (6), are satisfied with respect to some  $p_1$ .

**T3** (a). If  $\langle P, T, \mathbb{R}, v, m \rangle$  and  $\langle P, T, \mathbb{R}, v, m^* \rangle$  are measuring models for mass by collisions with respect to  $p_1$ , then  $m = m^*$ .

(b). If  $\langle P, T, \mathbb{R}, v, m \rangle$  and  $\langle P, T, \mathbb{R}, v, m^* \rangle$  are measuring models for mass-ratios by collisions then there is an  $\alpha \in \mathbb{R}^+$  such that for all  $p \in P$ :  $m(p) = \alpha \cdot m^*(p)$ .

## Measuring Models

In general, the idea of a measuring model comprises five features—all present in the previous example. First, the system captured by the measuring model satis-

<sup>15</sup>See Balzer & Muehlhoefer (1982), for a complete survey of all possible measuring models for mass in collision mechanics. The simplest case of collision along a straight line is avoided on purpose.

<sup>16</sup>In D4 and the corresponding proofs, I write  $w_i$ , for  $v(p_i, a) - v(p_i, b)$ ,  $i = 1, 2, 3$ , and  $u * s$  for  $\sum_{i \leq 3} u_i s_i$  where  $u = \langle u_1, u_2, u_3 \rangle$  and  $s = \langle s_1, s_2, s_3 \rangle$ .

fies some law—like the law of conservation of total momentum in D4 (a) (5). In general, we may suppose that the system under consideration gives rise to a structure  $x$  of the following form:

$$\langle D_1, \dots, D_k; A_1, \dots, A_m; R_1, \dots, R_n \rangle,$$

where  $D_1, \dots, D_k$  are sets of objects,  $A_1, \dots, A_m$  are sets of mathematical entities (like real numbers), and  $R_1, \dots, R_n$  are relations “over”  $D_1, \dots, D_k, A_1, \dots, A_m$ ,<sup>17</sup> that is,  $x$  is a structure of type  $\langle k, m, \sigma_1, \dots, \sigma_n \rangle$ . The law that holds in the given system may be expressed by some set-theoretic formula  $B(z_1, \dots, z_{k+m+n})$  containing at least the free variables  $z_1, \dots, z_{k+m+n}$  such that  $D_1, \dots, D_k, A_1, \dots, A_m, R_1, \dots, R_n$  are of the same types as  $z_1, \dots, z_{k+m+n}$  respectively. We then say that  $B$  has the type of  $x$  and  $B$ ’s validity in  $x$  then simply is stated by  $B(x)$  or  $B(D_1, \dots, R_n)$ .<sup>18</sup>

Second, in each measuring model, the function one wants to measure is uniquely determined by means of the other “parts” (functions and sets of objects) of the model and by the law that holds in the model. In the last example, this condition is expressed by T3 (a). It may be still further formalized by letting “ $B(P, T, \mathbb{R}, v, m)$ ” stand for “ $\langle P, \dots, m \rangle$  is a measuring model for mass by collisions.” That  $m$  is uniquely determined by  $P, T, \mathbb{R}, v$ , and statement  $B$  then amounts to<sup>19</sup> the following:

$$\forall m \forall m^* (B(P, \dots, v, m) \wedge B(P, \dots, v, m^*) \rightarrow m = m^*)^{20} \quad (1)$$

<sup>17</sup>For each  $r \in \mathbb{N}$ , we define syntactic  $r$ -types by induction: For each  $i \in \mathbb{N}$ ,  $i \leq r$ , we let  $[i]$  be an  $r$ -type, and, if  $\tau_1, \tau_2$  are  $r$ -types, then so are  $(\tau_1 \otimes \tau_2)$  and  $(\text{pow} \tau_1)$ . For each  $r$ -type,  $\tau$  and given sets  $S_1, \dots, S_r$ , we define the *echelon set*  $\tau(S_1, \dots, S_r)$  by induction. If  $\tau$  is some  $[i]$ , then  $\tau(S_1, \dots, S_r)$  is  $S_i$ , and, if  $\tau_j(S_1, \dots, S_r)$  is already defined for  $j = 1, 2$ , then  $(\tau_1 \otimes \tau_2)(S_1, \dots, S_r)$  is  $\tau_1(S_1, \dots, S_r) \times \tau_2(S_1, \dots, S_r)$ , and  $(\text{pow} \tau_1)(S_1, \dots, S_r)$  is  $\text{PO}(\tau_1(S_1, \dots, S_r))$ .  $R$  is a relation over  $S_1, \dots, S_r$ , iff there is some  $r$ -type  $\tau$  such that  $R \subseteq (\text{pow} \tau)(S_1, \dots, S_r)$ .  $x$  is a structure of type  $\langle k, m, \sigma_1, \dots, \sigma_n \rangle$ , iff  $k, m \in \mathbb{N}$ ,  $n \geq 1$ ,  $\sigma_1, \dots, \sigma_n$  are  $(k+m)$ -types, and  $x = \langle D_1, \dots, D_k; A_1, \dots, A_m; R_1, \dots, R_n \rangle$  where  $D_1, \dots, D_k$  are sets,  $A_1, \dots, A_m$  are sets of mathematical objects, and each  $R_i$  is a relation over  $D_1, \dots, D_k, A_1, \dots, A_m$ .

These definitions originally are found—with small deviations and in different notation—in Bourbaki (1968, pp. 259). In the following, for  $x = \langle D_1, \dots, D_k; A_1, \dots, A_m; R_1, \dots, R_n \rangle$ , and  $i \leq n$ , we write  $x_{-i}$  for the result of omitting  $R_i$  in  $x$ , and  $x_{-i}[R]$  for the result of substituting  $R_i$  in  $x$  by  $R$  (provided always that  $R_i$  and  $R$  are of the same  $(k+m)$ -type). Also we write  $R_i^x$  in order to denote the  $i$ th relation  $R_i$  occurring in  $x$ .

<sup>18</sup>Allowing for higher-order relations  $R$  covers those cases that in first-order formulations involve infinitely many axioms or that in infinitary logic involve infinitely long “formulas.” The distinction between “proper objects” and “mathematical entities” represented by  $D_1, \dots, D_k$  and  $A_1, \dots, A_m$  respectively is taken from Bourbaki and is mainly of practical importance here. Compare Balzer (1985b, chap. II).

<sup>19</sup> $P, T, \mathbb{R}$ , and  $v$  may be set theoretically regarded as free variables.

<sup>20</sup>Note that, in general, this condition is quite different from the one stating that  $m$  is a function that is formalized by  $\forall p, p' \in P (p = p' \rightarrow m(p) = m(p'))$ , or, more explicitly  $(1^*) \forall p, p' \in P \forall \alpha, \beta \in \mathbb{R} (\langle p, \alpha \rangle \in m \wedge \langle p', \beta \rangle \in m \wedge p = p' \rightarrow \alpha = \beta)$ . It is only in the “limit case” of  $P$  being a singleton that (1) follows from  $(1^*)$ . Intuitively (1) states the uniqueness of a set (of pairs of arguments and function values), whereas  $(1^*)$  states the

In the general notation used previously, uniqueness of  $R_i$  in structure  $x = \langle D_1, \dots, R_n \rangle$  is expressed as follows:

$$\forall R^*(B(D_1, \dots, R_n) \wedge B(D_1, \dots, R_{i-1}, R^*, R_{i+1}, \dots, R_n) \rightarrow R_i = R^*). \quad (2)$$

If it is relation  $R_i$  (which I always take to be a function, if necessary by switching to the characteristic function) that is uniquely determined in the measuring model  $x$ , I will say that  $x$  is a measuring model *for the  $i$ -th relation* ( $i \leq n$ ).

A third feature common to measuring models is that the function to be measured is determined uniquely only up to certain transformations by the law involved. In the example of D4 (a) above, one particle was required to have unit mass, D4 (a) (7). Usually, such requirements that refer to units or origins simply are omitted in the description of a measuring model with the effect that the function under consideration no longer is uniquely determined. The effect for uniqueness of omitting reference to units can be accounted for by weakening uniqueness to uniqueness-up-to-transformations-of-scale of a predetermined type. In the example of D4, the effect for uniqueness of dropping reference to a unit-mass, D4 (a) (7), is that the mass function  $m$  is no longer uniquely determined but is determined only up to a positive real number, T3 (b). The measuring model for mass-ratios by collision, D4 (b), thus, obtained is more realistic with respect to scientific practice, because it also covers those cases in which the process of measurement—the process of collision—does not involve a unit mass. This situation is acceptable, because still something reasonably accessible is uniquely determined: mass-ratios. If mass-ratios can be determined from velocities, one may eventually obtain “absolute” mass values by measuring the mass of one of the particles involved by means of another measuring model, different from the one at hand.

Here the general problem arises of what to regard as the right transformation of scale. There are quite a number of different, “established” transformations to be found in the literature. Obviously, the more general transformations for a function we admit, the less determination of that function we can achieve in a measuring model. For the time being, I will restrict considerations to linear transformations (more on this below). If  $f: D \rightarrow N$ , and  $f': D' \rightarrow N'$ , we say that  $f'$  is obtained from  $f$  by a linear transformation<sup>21</sup> (abbreviated by  $f' \approx f$ ), iff either ( $D = D'$ ,  $N = N'$ , and there exist  $\alpha, \beta \in \mathbb{R}$ ,  $\alpha > 0$  such that, for all  $a \in D$ :  $f'(a) = \alpha \cdot f(a) + \beta$ ) or  $f = f'$ . The weaker condition of uniqueness that results from (2) above, thus, is the following:

$$\forall R^*(B(D_1, \dots, R_n) \wedge B(D_1, \dots, R_{i-1}, R^*, R_{i+1}, \dots, R_n) \rightarrow R_i \approx R^*) \quad (3)$$

uniqueness of function values—which may be construed as components of elements of the sets occurring in (1).

<sup>21</sup>This definition can be easily extended to functions taking values in vector spaces over ordered fields.

That is, if  $\langle D_1, \dots, R_n \rangle$  is a measuring model for the  $i$ -th relation, and, if, by replacing  $R_i$  by some  $R^*$ , we still have a measuring model for the  $i$ -th relation, then  $R^*$  is obtained from  $R_i$  by a linear transformation. In other words, the function  $R_i$  to be determined in measuring model  $\langle D_1, \dots, R_n \rangle$  is determined by the law  $B$  and by the other "parts"  $D_1, \dots, R_{i-1}, R_{i+1}, R_n$  of the measuring model up to a linear transformation.

The fourth feature inherent in measuring models is that the measured value, or more generally, the function of which this value is a function value, can be *computed* from the other "parts" of the measuring model. This means that, for each argument  $a$ , the function value  $R_i(a)$  can be computed from appropriate values of the other functions  $R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n$ . The computation has to start with a finite input; thus, only finitely many values  $R_j(a_s)$  ( $j = 1, \dots, n, j \neq i, s \leq t$  for some  $t \in \mathbb{N}$ ) will be needed. That is, the computation of the measured value does not really use "all of" the other functions but only some finite parts of them. This idea may be formalized by means of a rather general notion of a substructure. If  $x = \langle D_1, \dots, D_k; A_1, \dots, A_m; R_1, \dots, R_n \rangle$  is a structure of type  $\tau = \langle k, m, \sigma_1, \dots, \sigma_n \rangle$ , then  $z = \langle D'_1, \dots, D'_k; A'_1, \dots, A'_m; R'_1, \dots, R'_n \rangle$  is a *substructure* of  $x$  iff for all  $i \leq k$ :  $D'_i \subseteq D_i$ ; for all  $i \leq m$ :  $A'_i \subseteq A_i$ ; and for all  $i \leq n$ :  $R'_i \subseteq R_i$ , and  $z$  again is a structure of type  $\tau$ . The requirement of computability of the measured value in a measuring model  $x = \langle D_1, \dots, R_n \rangle$  for the  $i$ -th relation then can be stated as follows. For each  $a$  in the domain of  $R_i$ , there is a finite substructure  $z_a$  of  $x_i$  (see footnote 17) such that  $R_i(a)$  is computable from  $z_a$ . In order to link this notion to the standard notion of computability, some further encoding of  $R_i(a)$  and of  $z_a$  will be necessary, but it will depend on the particular case, and not much can be said about it in general.

Usually the computations arise in a natural way from the law  $B$  that governs the measuring model. Often this law consists of a simple equation that can be solved for the measured value  $R_i(a)$  (as in the example of weight measurement following). In other cases, the computation may involve some mathematics. The computation of mass values in the case of the example from D4 involves linear algebra in order to solve the system of linear equations given by the basic law (see the proof of T4).

A final feature of measuring models is that the measured value in fact depends on the other parts of the model. In the conceptual frame used here, function  $R_i$  might be defined in a purely mathematical way. A real valued function  $R_i: D_1 \rightarrow \mathbb{R}$  could be defined, for example, by:  $R_i(a) = 5$  for all  $a \in D$ . Such an  $R_i$  would satisfy the requirement of uniqueness, but the measured value  $R_i(a)$  would be mathematically defined, and there would be no need for measurement. Such cases are excluded by requiring that  $R_i$  in fact changes when the other parts of the model change. Here topological concepts come into play. Consider the following relation  $\theta_B$  given by the formula  $B$  that characterizes a measuring model:  $\theta_B(u, v)$ , iff there is  $x$  such that  $B(x)$ ,  $u = x_{-i}$  and  $v = R_i^x$ . Factorizing  $\theta_B$  modulo  $\approx$  in its second argument yields a relation  $\theta_B^\approx$  given by  $\theta_B^\approx(u, v)$  iff there is  $x$  such that  $B(x)$ ,  $u = x_{-i}$  and  $R_i^x \in v$ . If  $B$  determines  $R_i$  in  $x$  up to linear transformations,

then  $\theta_B^\approx$  will be a function. Now purely mathematical definitions of  $R_i$  as the one mentioned above might be excluded just by saying that  $\theta_B^\approx$  is not constant. Still, this seems to leave room for *ad hoc* definitions that really do not involve measurement. A stronger and more adequate requirement is that  $\theta_B^\approx$  be a continuous, nonconstant function, continuity being defined relative to some natural topologies introduced on the domain and range of  $\theta_B^\approx$  by a given topology on the class  $\{x/B(x)\}$  of all measuring models given by  $B$ . In the example of D4, such topologies are given by the distances of the function values  $|v^x(p, a) - v^z(p, a)|$  and  $|m^x(p) - m^z(p)|$  for the  $v$ - and  $m$ -functions in different models  $x, z$  (in which the sets  $P$  and  $T$  are the same).

Summing up these features, we obtain the following:

- D5** (a).  $x$  is a measuring model (for the  $i$ -th relation) characterized by  $B$  and  $\psi$  iff there exist  $D_1, \dots, D_k, A_1, \dots, A_m, R_1, \dots, R_n$  such that  $x = \langle D_1, \dots, D_k, A_1, \dots, A_m, R_1, \dots, R_n \rangle$  and the following holds:
1.  $i \leq n$ .
  2.  $B$  is a set theoretic statement of a law;  $B$  is of the type of  $x$  and  $B$  is valid in  $x$ .
  3.  $\psi$  is a topology on  $\{x/B(x)\}$ .
  4. For all  $R^*$ : if  $B(D_1, \dots, R_{i-1}, R^*, R_{i+1}, \dots, R_n)$ , then  $R_i \approx R^*$ .
  5. For all  $a$  in the domain of  $R$ , there is a finite substructure  $z_a$  of  $\langle D_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n \rangle$  such that  $R_i(a)$  is computable "up to  $\approx$ " from  $z_a$  (after appropriate encoding).
  6. relation  $\theta_B^\approx$  is a continuous function (relative to suitable topologies induced by  $\psi$ ) and not constant.
- (b).  $x$  is a measuring model (for the  $i$ -th relation) iff there exist  $B$  and  $\psi$  such that  $x$  is a measuring model (for the  $i$ -th relation) characterized by  $B$  and  $\psi$ .

## Methods of Measurement

As before, consideration of a single system is unlikely to lead us to a measuring model, because it will hardly reveal law-like features as required in D5 (a) (2). In reality a large number of real systems has to be studied and compared with each other so that a regularity can be detected and a law formulated correspondingly. A more natural basis for the definition of measuring models, therefore, is given by a class of systems or, after conceptualization, by a class of structures.

- D6**  $\mathcal{L}$  is a method of measurement (for the  $i$ -th relation) iff there exist  $B$  and  $\psi$  such that the following holds:

1.  $B$  is a set theoretic formula that can be interpreted in structures of the form

$\langle D_1, \dots, D_k; A_1, \dots, A_m; R_1, \dots, R_n \rangle$ ,  $i \leq n$ , and  $B$  expresses a law.

2.  $\mathcal{L}$  is the class of all structures in which  $B$  is valid.
3. Each  $x \in B$  is a measuring model for the  $i$ -th relation characterized by  $B$  and  $\psi$ .
4.  $\mathcal{L}$  contains (descriptions of) many real systems.

D6 (1) and (3) just rephrase D5. D6 (4) is intended to exclude contrived, abstract examples. We speak of a method here in the following, derived sense. For each proper method of measurement, we can think of the class of all cases in which the method might be applied successfully. Each such case would yield a real system, given either by the process of applying the method or by the result of having the method applied. In this way, each method of measurement corresponds to a class of systems, and we decide to call this class itself a method of measurement.

Obviously the requirements of  $B$  being a law and of  $\mathcal{L}$  containing many real systems cannot be formalized. They tie the notion of a method of measurement to pragmatics and to "reality." What is a law can only be made out on the basis of theoretically minded human interaction, and what is a (description of a) real system can be made out only by some reference to experience. We cannot expect the notion of a method of measurement to be definable more formally than the notion of an empirical theory, simply because we want methods of measurement to cover cases governed by proper theoretical laws (which occur in established theories) as well as cases of fundamental measurement in which the law characterizing the measuring models is not a law of any established theory.

### Examples

1. The classes  $\mathcal{L}_1$  and  $\mathcal{L}_2$  of all measuring models for mass and mass-ratios by collisions are methods of measurement (for the second relation).

2. If we rearrange the components in models of length measurement like this:  $\langle D; \mathbb{R}; \leq, +, \lesssim, o, \omega \rangle$ , the class of all models of length measurement yields a method of measurement  $\mathcal{L}_3$  for the fifth relation.

3. Let  $\mathcal{L}_4$  be the class of all structures  $\langle P; T, \mathbb{R}; s, m, f_1, \dots, f_n \rangle$  for which there exist  $p_1, p_2, p_3$  ("earth," "particle the weight of which is measured," "particle marking the origin"), all distinct, such that the following: (a)  $P = \{p_1, p_2, p_3\}$ ; (b)  $T \subseteq \mathbb{R}$  is an open interval (set of "instants"); (c)  $s: P \times T \rightarrow \mathbb{R}^3$  is smooth ("position function"); (d)  $m: P \rightarrow \mathbb{R}^+$  ("mass function"); (e) for all  $i \leq n$ :  $f_i: P \times T \rightarrow \mathbb{R}^3$  ("force functions"); (f) there is  $k \in \mathbb{R}^+$  such that, for all  $t \in T$ :  $f_1(p_2, t) = -k(s(p_3, t) - s(p_2, t))$  ("Hooke's law"); (g) for all  $t \in T$ :  $f_1(p_2, t) = -f_2(p_2, t)$  ("actio-reactio law," specialized to  $p_2$ ); (h) for all  $t \in T$ :  $f_2(p_1, t) = f_2(p_3, t) = 0$ . Elements of  $\mathcal{L}_4$  capture measurements of weight ( $= f_2(p_2, t)$ ) under



the following interpretation:<sup>22</sup>  $f_1$  denotes gravitational force and  $f_2$  Hooke's force exerted in a system in which  $p_2$  (some object) is hung on a spring balance (where  $p_3$  marks the end of the spring in the unextended case). Clearly function  $f_2$  is uniquely determined in members of  $\mathcal{L}_4$ , and  $\mathcal{L}_4$  is a method of measurement (for the fourth relation).

Note that the class of all closed extensive positive structures would not be a good candidate for a method of measurement because of D6 (4). There are no real systems that can be described in terms of D4.

Unlike the notion of fundamental measurement, the notions of a measuring model and of a method of measurement are neutral with respect to the structure of science. They fit in naturally into the picture of science as a web or a net of structures, models, or theories. Measuring models can be treated on a par with models of theories, and methods of measurement on a par with classes of models of theories. According to the most sophisticated notion available,<sup>23</sup> one main constituent of an empirical theory is a net consisting of a class of models and of various subclasses, ("specializations"). Methods of measurement are just a particular kind of such specializations. In this picture, no priority is given to "direct observation" or to distinguished forms of measurement.

## COMPARISON

Having outlined the basic features of the representationalist and the structuralist view of measurement, we now can compare the two.

First, let us compare the notions of a fundamental measurement class and of a method of measurement as given by D2 and D6. A fundamental measurement class is a class of systems of fundamental measurement; that is, a class of structures characterized by certain axioms and so is a method of measurement (D2 (2) and D6 (2)). The axioms for empirical structures in systems of fundamental measurement have to be grounded in empirical regularities and so have the axioms characterizing the measuring models D2 (1) and D6 (1). In systems of fundamental measurement, there is the representing homomorphism  $\omega$  that is determined uniquely up to specified transformations. In measuring models, there is the corresponding requirement that the function to be measured be determined uniquely up to some specified transformation D2 (3.3) and D6 (3). The requirements of computability and of proper dependence inherent in D6 (3) usually also are satisfied in systems of fundamental measurement—although not required explicitly.

Besides these correspondences, there are the following differences. Systems

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<sup>22</sup>See Balzer and Moulines (1980).

<sup>23</sup>(Balzer, Moulines, & Sneed, 1987, chap. IV).

of fundamental measurement have an inner structure more specific than that of measuring models. They have to consist of two "smaller" structures (the empirical and the mathematical one) and a homomorphism between those. No such requirement is present for measuring models. In this respect, measuring models are much more general. Furthermore, the empirical relations in systems of fundamental measurement have to represent concrete operations, whereas, on the side of measuring models, no distinction among the relations  $R_1, \dots, R_n$  (like observational-theoretical or concrete-abstract) is required. In this respect also, measuring models are more general.

This indicates that methods of measurement are more general than systems of fundamental measurement. There is, however, a little difficulty. As noted previously, the idealized systems of fundamental measurement do not describe real systems, and if we consider corresponding classes of such systems as candidates for methods of measurement, usually D6 (4) will not be satisfied. There are two remarks to that point. First, one might simply drop D6 (4) and, thus, eliminate the difficulty. This might be justified by saying that, for the sake of comparison, we are primarily interested in the conceptual structure of measuring models. I do not regard this as an adequate reaction. Rather I would remark secondly that the transition from the representationalist to the structuralist view is not completely smooth. It seems to involve a kind of "Kuhn-loss," and what is lost are precisely those superidealized structures that have no real applications. As soon as approximation is brought into play, an analogue to approximative reduction perhaps would be appropriate to describe the relation between the two approaches.

A second dimension of comparison is that of implications for the structure of science. It seems that the structuralist view fits better with the actual, overall structure of empirical science than does the received view that has some difficulties "getting off the ground" of empirical structures and systems of fundamental measurement. We may say that the structuralist view frees philosophy of science from dogmatic assumptions of logical empiricism without loss of potential of drawing a fine-grained picture of science.<sup>24</sup>

Finally it has to be stressed that the structuralist view covers both kinds of representationalist measurement: fundamental as well as derived (or conjoint) measurement. It does so by substantial generalization. In connection with fundamental measurement, one result of this generalization is that the distinction between a qualitative ("empirical") and a quantitative (numerical) level vanishes. In connection with derived measurement, the greater generality of measuring models is obvious. Derived measurement requires a definition, whereas, in measuring models, only some law-like connection is required. The law  $B$  in a measuring model need not be suited as a general definition of the term to be measured.

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<sup>24</sup>It is not difficult, for instance, to draw the hierarchical picture of science within the structuralist view.

If we use the term "extension" in a broad way that does not allow "parts" of the representationalist account to be properly included in the extended account but to be only captured by the structuralist view in the limit, then these considerations may be summarized by stating that the structuralist view is a proper and essential extension of the representationalist view of measurement.

## PERSPECTIVES AND PROBLEMS

In the section, *The Structuralist View of Measurement*, only the most basic features of the structuralist view were sketched. When implemented in the semantic or the structuralist approach to theories, the structuralist view of measurement opens up a wide range of applications. In this final section, I want at least briefly to touch on some achievements, possibilities, and problems of this view.

Its most definite achievement consists in the development of a formal definition of theoreticity that yields a formal distinction between *T*-theoretical and *T*-nontheoretical terms in any axiomatized theory *T*. The definition evolved from investigations of measurement under the structuralist view, and it yields the expected distinctions when applied to axiomatized versions of real-life theories. Moreover, this definition for the first time opens a way for empirical and precise studies of the global structure of science—in contrast to logical empiricism or the received view that both simply postulate and presuppose a hierarchical structure. The new definition of theoreticity being available in the literature, there is no need to go into the details here (see footnote 12).

A second achievement consists in complete accounts of complicated measurements involving several intermediate steps (preliminary measured values) and several different theories to account for these steps. A simple example is provided by the measurement of mass by collisions: In order to obtain the desired mass-values, we have to measure velocities, that is, intervals of space and time. So the whole process of measurement may be divided into several subprocesses: one for each measurement of a distance in space or time, and one consisting of the whole process. In total, three different kinds of phenomena and three corresponding theories are involved: phenomena of spatial distance, of distance in time, and of the inertial behavior of moving particles. The masses are calculated from "presupposed" values of velocities by means of collision mechanics, whereas the presupposed velocity values themselves are calculated from other, "presupposed" values of distances and times, which in turn are measured by means of geometric and chronometric devices (i.e., by means involving geometry and chronometry). A full analysis yields a whole chain of values obtained from iterated calculations of measured values from presupposed values.

The idea of a chain of values is not very original. Furthermore, it does not give a satisfactory account of the respective measurement, for it does not contain any information about the theories involved. Let me call a description of some

process of measurement *complete*, only if it gives the relevant sequence of measured values plus the equations (theories) that are applied in each step of the calculation. Again this idea of a complete description does not seem to be very original. However, it did not appear in the literature up to now, and mainly so for one reason (now becoming obvious): a prejudiced view of measurement that dissociates measurement too much from theory. Within the structuralist view, complete descriptions of measurement can be given easily. The concept of a *chain of measurements* and of a *measurement graph* (a sequence, respectively a graph of interrelated measuring models) provide simple and powerful descriptive tools.<sup>25</sup>

It may be objected here that there is a still more strict notion of a complete description capturing also errors of measurement and experimental design. The important topic of errors of measurement has not yet been addressed from the structuralist point of view. It seems possible, however, to reformulate existing accounts in terms of measuring models and substructures. Such a reformulation might prove helpful, especially in cases of measurement in highly developed theories that may involve features of theory dynamics and of approximate comparison of theories. The subtle picture of how errors of measurement come up together with theory provided by H. E. Kyburg Jr.<sup>26</sup> is of particular relevance here. Furthermore, some general structuralist apparatus might be helpful.<sup>27</sup> As far as experimental design is concerned, I do not think that this can be included in a systematic description of measurement. All the *systematic* features are somehow captured by the theories governing the process of measurement, any further features of special design being attached to special cases and in this sense being unsystematic.

A third achievement of the structuralist view—closely related to the second one—is that it treats measurement and theory in the same model-theoretic way. The basic units of analysis are systems (represented by models), and investigations aim at revealing their inner structure (represented by types and axioms) as well as their outer relations to other systems. In addition to “mere” models, measuring models are distinguished by their uniqueness properties. However, this does not prevent them from functioning in the network of theories much like ordinary models. This perspective is likely to prevent us from ascribing to measured values a *preferred* status of given, nonhypothetical items against which theories are tested, and it yields a better starting point for the investigation and understanding of how theory and measurement function and develop together. So much has become clear by now: The development and test of comprehensive theoretical networks do not fit into the simple picture of collecting independent

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<sup>25</sup>See (Balzer, 1985b, chap. IV) and (Balzer & Wollmershaeuser, 1986).

<sup>26</sup>See (Kyburg, 1984) and his contribution to this volume. Although his two rules may be still controversial, the basic picture he provides is not.

<sup>27</sup>See (Balzer et al. 1987, chap. VII).

data and then theoretically organizing them. There is much interplay between the development of theoretical pictures and what counts as data in a given state of the development. This interplay can be structured best when theory and data (measurement) are represented in the same way, as having equal rights.

This brings me to some further topics not yet worked out, for the treatment of which the structuralist view seems to be promising. The first topic is theory development including features of measurement, test, and approximation. Up to now, pictures of theory development do not capture aspects of measurement, and if they include features of approximation at all, they do so (see footnote 27) just by blurring the idealized theoretical pictures. However, the real source of such fuzziness is of course the impossibility of exactly measuring numerical functions. The inclusion of measurement into the diachronic picture of theoretical development, therefore, not only will yield an explanation of why theories are only approximately true; it also will provide a frame for depicting the origins and changes of the degrees of accuracy relevant for each application of a theory at a given time.

A second promising topic is confirmation. There are the classical ideas about a choice between alternative hypotheses attached to the names of Bacon, Nicod, and Popper, and the ideas of Carnap and Hempel about degrees of confirmation of a hypothesis (theory) in a given structure. However, these ideas are intended to apply to one isolated hypothesis that is matched against an independently given data-base and given alternative hypotheses. Philosophers abandoned these ideas as inadequate for more comprehensive arrays of theory long ago, but only recently did some positive move occur, in the form of the "bootstrap view" of confirmation Glymour (1980) has put forward. Although this account in fact hits the central pattern underlying confirmation in large theoretical networks, it falls short of providing a global analogue to the idea of a degree of confirmation. The reason for this limitation again is the conceptual separation of measurement from theory. Glymour represented measurement by "computations," sequences of presupposed and measured values, the theories underlying the various steps being ignored. Yet as soon as theories are properly included in the picture—which means that measurement is treated within the structuralist view—the notion of a degree of confirmation for more comprehensive theoretical arrays can be attacked once again by combining bootstrap ideas with those of partial implication.<sup>28</sup>

A third promising topic is reference. Up to now, reference has been approached merely from the side of philosophy and philosophy of language. The structuralist view of measurement yields an elegant conceptual frame for defining the referents of terms of scientific, extensional theories. This may form the basis for a precise, detailed theory of reference for empirical, extensional theories, that is, for a theory of reference suited for the philosophy of science. For example,

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<sup>28</sup>For the purely theoretical side of the issue that also is one-sided by neglecting measurement, see (Balzer et al., 1987, chap. VIII).

the referent of a numerical function in a theory is given as that assignment of numbers to real objects (which are represented by appropriate arguments in models of real systems) that is determined by the theoretical axioms, constraints, and interrelations as well as by the real systems to which the theory applies. The fact that, among the real systems, there usually are many measuring models for the function in question will often guarantee a unique referent, whereas the theoretical constraints provide for coherence.<sup>29</sup>

I want to close by pointing to a problem central to the present account, namely that of drawing a demarcation between “mere” models of a theory and measuring models for some term of the theory. The crucial condition distinguishing measuring models from mere models is that of uniqueness of the function to be measured. However, as already noted, to require strict uniqueness would deprive the notion of a measuring model of practical interest, because usually the theoretical laws alone do not imply such strict uniqueness. An adequate concept of a measuring model has to be formulated with a requirement of uniqueness up to some kind of transformation. There are different kinds of transformations one may consider; thus, the problem is to choose the “right” kind of transformation to be employed in the definition of a measuring model. This really is a problem of demarcation, because, if the transformations chosen are too general, we will not be able to determine the function considered in any interesting sense, and, therefore, we will not have a *measuring* model before us. If the transformations are chosen very narrowly, we may have a measuring model but one that is not interesting in connection with its corresponding theory, because it applies only to few and uninteresting real systems.

In the section, Structuralist View of Measurement, this problem was “solved”, or rather suppressed, by pretending an absolutely adequate class of transformations, adequate for all numerical functions—linear transformations. However, inspection of examples shows that each theory has its own adequate class of transformations. When determining distances in affine geometry, only dilatations are meaningful, whereas, in a fundamental determination of temperature, the full class of linear transformations is appropriate. In a determination of positions by means of classical mechanics, Galilei transformations might be suggested, because such transformations connect different frames of reference. In utility theory, the determination of utility in standard approaches<sup>30</sup> proceeds

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<sup>29</sup>Compare (Balzer, 1985b, Sec. 23, 1987), and (Balzer, Lauth, Zoubek, 1989).

<sup>30</sup>By “standard approaches,” I here mean those exemplified in traditional economic literature like, (Henderson & Quandt, 1958). There is another approach originating from v. Neumann & Morgenstern (1947) regarding utilities as determined by preferences and probabilities (“beliefs”) up to linear transformations that now has started to penetrate “standard” economic literature. Still a third view, represented by (Jeffrey, 1965), is worth mentioning (in order to show that the issue really is not settled) according to which utility together with probability are determined by means of preferences up to so-called “Goedel–Bolker” transformations.

only up to monotone transformations: By such procedures, only an ordering of the alternatives can be achieved, and the numerical values for utility are really not determined at all. From these considerations, one is tempted to jump to the conclusion that each theory for each of its terms determines the appropriate class of transformations to be used in the definition of the measuring models for that term. However, this view deprives the condition of uniqueness of any content. Each  $R_i$  is unique up to natural transformations, if by natural transformations we mean those that preserve the property of being a measuring model. Uniqueness is central to measurement proper; thus, a criterion is necessary that prevents the condition of uniqueness from becoming trivial.

There are two lines of attacking the problem. First, formal considerations may be used to exclude certain kinds of transformations or invariances as inadmissible for a given method of measurement. Recently Luce and Narens<sup>31</sup> achieved progress here. By introducing the concept of a scale type as consisting of a degree of uniqueness and a degree of homogeneity, they were able to show that, under special conditions (like the presence of a concatenation operation or the existence of some representation onto the reals), only certain, few scale types are possible.

Second, by concentrating on the "positive", real-life cases of measurement up to certain transformations of scale, those cases in which no doubt is likely to occur, we see that there the transformations are completely determined by finitely many—actually very few—parameters, like two numbers in the case of linear transformations. Moreover, giving some particular value to any of these parameters has a straightforward, empirical meaning like choosing a particular object as unit or as origin. We, therefore, might say that some class of transformations is *admissible*, iff it is determined by finitely many parameters, and if the choice of each parameter corresponds to a choice of some basic object in the model, that is, of some element of the base sets  $D_1, \dots, D_k$  of the model. Of course the meaning of "corresponds" here needs further elaboration, and a little reflection shows that a kind of reduction to qualitative comparison lurks behind it—much like that underlying the representationalist view.

## APPENDIX

*Proof of T1:* (a) Let  $b \in U$ ,  $b' \in U'$ . By D1 (a) (6), there are  $b_1, \dots, b_n \in U$  such that  $b_1 \circ \dots \circ b_n = b'$ . Suppose  $n \geq 1$ . Then, for all  $i \leq n$ :  $b_i < b'$  by D1 (a) (4) and (2). But also, by D1 (a) (6), there are  $b'_1, \dots, b'_m \in U'$  such that  $b'_1 \circ \dots \circ b'_m = b$ , which implies  $b'_j \leq b$  by D1 (a) (2) and (4). So  $b'_1 \leq b = b_i < b' = b'_1$  by D1 (a) (5), (i.e.,  $b'_1 < b'_1$ , which yields a contradiction with D1 (a) (2). So  $n = 1$ , (i.e.,  $b = b_1 = b'$ ), from which we obtain  $b = b' \#$ .  
(b) Let  $x$  be a model with units  $U$  and  $x'$  a model with units  $U'$ .

<sup>31</sup>Compare (Luce & Narens, 1985) and (Narens, 1985).

*Lemma 1* For all  $b_1, \dots, b_n, b'_1, \dots, b'_m \in U$ :

$$b_1 \circ \dots \circ b_n \equiv b'_1 \circ \dots \circ b'_m \text{ iff } n = m.$$

*Proof of the Lemma:* If  $n = m$ , the left-hand side follows from D1 (a) (2), (4), and (5) by induction with respect to  $n$ . Conversely, if without loss of generality  $m < n$ , then  $b'_1 \circ \dots \circ b'_m \equiv b_1 \circ \dots \circ b_m < (b_1 \circ \dots \circ b_m) \circ (b_{m+1} \circ \dots \circ b_n) \equiv b_1 \circ \dots \circ b_n$ , by the first part of the proof, and by D1 (a) (2) and (4)#.

Now let  $b \in U$ ,  $\omega(b) = \beta$ ,  $\omega'(b) = \tau$ , and  $\alpha := \beta/\tau > 0$ . Let  $a \in D$ . By D1 (a) (5), there exist  $b_1, \dots, b_n \in U$  and  $b'_1, \dots, b'_m \in U'$  such that (1)  $b_1 \circ \dots \circ b_n \equiv a \equiv b'_1 \circ \dots \circ b'_m$ . By D1 (a) (7.1) and part (a):  $\omega(b_i) = \omega(b'_i)$  and  $\omega'(b_i) = \omega'(b'_i)$  for all  $i \leq n$  and  $j \leq m$ . This, together with (1) and D1 (a) (7.2), yields  $\omega(a) = n\beta = n\omega(b_1)$  and  $\omega'(a) = m\omega'(b) = m\tau$ .

Applying Lemma 1 to  $U \cup U'$ , we obtain from (1):  $n = m$ , so  $\omega(a) = n\beta = n\alpha\tau = \alpha n\tau = \alpha\omega'(a)$ #.

(c) Define  $\omega$  by  $\omega(b) = 1$  for all  $b \in U$  and  $\omega(a) = n$ , iff  $a \equiv b_1 \circ \dots \circ b_n$ , according to D1 (a) (6). Then D1 (a) (7) follows from D1 (a) (6) and Lemma 1, and uniqueness follows from (b) #.

*Proof of T2:* (a) See (Krantz et al., 1971), T1 p. 74 #.

(b) We first need Lemma 2.

*Lemma 2* If D3 (a) (1) to (4) hold, then  $\circ$  is commutative.

*Proof:* (Krantz et al., 1971), p. 78, lemma 3#.

D3 (a) (2) follows from (i) and D1 (a) (2). By (iii),  $\cup \circ$  is total, and associativity follows from (i) and D1 (a) (3). D3 (a) (4) follows from (i), (iii) and D1 (a) (4). In order to prove D3 (a) (5) let  $a, b, c, d \in D^{\cup X}$  such that  $a < b$ . If  $c \leq d$ , (5) holds for  $n = 1$  by D3 (a) (4). So let  $d < c$ . If  $a \circ c < b \circ d$ , (5) holds for  $n = 1$ . Otherwise  $a \circ c > b \circ d$  (by D3 (a) (1) to (4), which are already proved. By (i), there is an  $x \in X$  such that  $a, b, c, d \in D^x$ , and, by D1 (a) (6), there are  $a_1, \dots, a_{n(a)}, b_1, \dots, b_{n(b)}, c_1, \dots, c_{n(c)}, d_1, \dots, d_{n(d)} \in U^x$  such that  $a_1 \circ \dots \circ a_{n(a)} \equiv a$  and similarly for  $b, c, d$ . From  $a < b$ , it follows that  $a <^x b$ , and from Lemma 1, we obtain that  $n(a) < n(b)$ , i.e.  $b \equiv^x a_1 \circ \dots \circ a_{n(a)} \circ b_{n(a)+1} \circ \dots \circ b_{n(b)}$ , and similarly from  $d < c$ , we obtain  $n(d) < n(c)$ , i.e.  $c \equiv^x d_1 \circ \dots \circ d_{n(d)} \circ c_{n(d)+1} \circ \dots \circ c_{n(c)}$ . Now let  $c - d := c_{n(d)+1} \circ \dots \circ c_{n(c)}$  and  $b - a := b_{n(a)+1} \circ \dots \circ b_{n(b)}$ . From  $a \circ c > b \circ d$ , Lemma 2, and D1 (a) (4), we obtain that  $c - d > b - a$ . By (ii), there is an extension  $z \in X$  of  $x$  and some  $n \in \mathbb{N}$  such that  $n(b - a)$  is defined in  $z$ , and  $c - d \leq^z n(b - a)$ . By (iii) and (i), we find some extension  $u$  of  $z$  such that  $na$  and  $nb$  also are defined in  $u$ . From D3 (a) (1) to (4), Lemma 2, it follows, by straightforward calculation, that  $n(b - a) \equiv nb - na$  where the latter entity is constructed like  $b - a$  above. In  $u$ , we then have  $c - d \leq^u nb - na$  which, by Lemma 2, implies  $na \circ c \leq^u nb \circ d$ . But then  $na \circ c \leq nb \circ d$ , by the definition of  $\cup X$ #.

*Proof of T3:*

(b) Let  $m_i := m(p_i)$  for  $i = 1, 2, 3$ , Then D4 (a) (5) can be rewritten as  $\sum_{i \leq 3} m_i w_i$

(1)



$= 0$ . This is a homogeneous system of linear equations. Let  $W$  be the matrix  $(w_1, w_2, w_3)$ ,  $Z := \{(m_1, m_2, m_3) \in \mathbb{R}^3 / m_i > 0 \text{ for } 1 \leq i \leq 3\}$  and  $L := Z \cap \text{Kernel}(W)$ . By a theorem by Tucker (Tucker, 1956), Corollary 1A, D4 (a) (6) implies (2): The existence of a positive solution of (1). Now (1) has a unique (up to a positive factor) and positive solution iff  $\text{Dim}(L) = 1$ , but  $L$  is an open subset of  $\text{Kernel}(W)$ , so  $\text{Dim}(L) = \text{Dim}(\text{Kernel}(W))$ . Therefore, (1) has a unique and positive solution, iff  $\text{Dim}(\text{Kernel}(W)) = 1$ . By a well known theorem of linear algebra:  $\text{Dim}(\text{Kernel}(W)) + \text{Rank}(W) = 3$ . So (3): (1) has a unique positive solution, iff  $\text{Rank}(W) = 2$ . By D4 (a) (6), Part 1, we have  $\text{Rank}(W) = 2$ , and from (3) and (2), we obtain that the solution of (1) is unique up to a positive factor#.

Part (a) now follows from D4 (a) (7)#.

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